The second derivative

- What does the first derivative tell us?
  - $f' > 0$ where $f$ is increasing
  - $f' < 0$ where $f$ is decreasing
  - $f' = 0$ where the tangent line is horizontal.
  - $f'$ undefined, means there is a corner, pick, or the graph looks vertical.

- Concavities and derivatives
  - What is the behavior of the first derivative when a function is concave up?
  - What is the behavior of the first derivative when a function is concave down?

To answer these questions, look at the graph below. This graph displays a function and its derivative. Of course the function is in green and its derivative in blue. Why?
  - Where is the function concave up?
  - Where the function is concave up, is the first derivative increasing or decreasing?
  - Where is the function concave down?
  - Where the function is concave down, is the first derivative increasing or decreasing?
So far we have the following observations:

- Where a function is concave up, the first derivative is increasing.
- Where a function is concave down, the first derivative is decreasing.

The intervals where the first derivative increases it must happen that its derivative is positive, and where the first derivative is decreasing it must happen that its derivative is negative.

- **Notation**: The second derivative of a function \( y = f(x) \) is the derivative of the first derivative

\[
y = f(x), \quad y' = \frac{df}{dx}, \quad y'' = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2}
\]

From our previous observations we have:

1. A function being concave up is equivalent to:
   - Having first derivative increasing, \( f' \) increasing
   - Having the second derivative positive, \( f'' > 0 \)

2. A function being concave down is equivalent to:
   - Having first derivative decreasing, \( f' \) decreasing
   - Having negative second derivative, \( f'' < 0 \)

Let’s look at the graph of \( f''(x) \) to verify those characteristics of \( f(x) \)

- What does the second derivative tell us? Remember that we are talking only about points in the domain of the function. Even more the points of the domain where the function is continuous. Hence,
\[ f'' > 0 \], \( f' \) is increasing, \( f \) is concave up
\[ f'' < 0 \], \( f' \) is decreasing, \( f \) is concave down.
\[ f'' = 0 \], \( f' \) has a horizontal tangent line
\[ f'(x) \) undefined. It means, the function is defined there and the graph, locally, looks like one of the following:

Notes:

1. The points where the function changes concavities are the points where the \( f' \) changes from decreasing to increasing or from increasing to decreasing. Those points are called **INFLECTION POINTS**. At those points the second derivative is either zero or undefined.

2. The **candidates** to be inflection points are the critical points of the first derivative which are in the domain of the function, it is the points in the domain where \( f''(x) = 0 \), or, \( f''(x) \) is undefined. However, the same way as with the first derivative, to guarantee they are inflection points one needs to show the change of concavity, which amounts to showing those points are either local max or local.
min of the first derivative. We can use the first derivative test applied to the function \( f' \). It amounts to:

a. Locating on the number line the candidates to be inflection points.

b. Locating the points which are not in the domain of \( f(x) \)

c. Determining the sign of the second derivative on each interval. If there is a change of sign of the second derivative at the candidates to be inflection points, then they become inflection points. Where the second derivative is positive the function is concave up, and where it is negative the function is concave down.

**EXAMPLE 1**

You can work out this example in with technology.

Consider the function \( f(x) = x^2 + \frac{1}{x} \). It is given that \( f''(x) = 2 + \frac{2}{x^3} \) to determine the inflection points we follow the steps described above:

- Candidates to be inflection points, \( f''(x) = 0 \) when \( x = -1 \), and \( f''(x) \) is undefined at \( x = 0 \). Since \( x = 0 \) is not in the domain, the only candidate to be an inflection point is \( x = -1 \).

- On the number line we have the following intervals \((-\infty, -1), (-1, 0), (0, \infty)\)

- Table of values

<table>
<thead>
<tr>
<th>Test Points</th>
<th>((-\infty, -1))</th>
<th>((-1, 0))</th>
<th>((0, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of ( f'' )</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Concavity</td>
<td>( \cap )</td>
<td>( \cup )</td>
<td>( \cap )</td>
</tr>
</tbody>
</table>

Obtain the graph of the function using Mathematica to double check these results.

**EXERCISE 1**

Complete the information in the table

\[
\begin{array}{c|c|c|c}
\hline
f(x) & -0.5 & 0 & 0.5 \\
\hline
f > 0 & -0.3 & 0 & 0.3 \\
\hline
f' < 0 & -0.2 & 0 & 0.2 \\
\hline
f'' > 0 & -0.1 & 0 & 0.1 \\
\hline
\end{array}
\]
EXERCISE 2

The following graphs contain the original function, its first and second derivatives. Identify which one is which and justify your answer.

EXERCISE 3

1. Let $y = x^3 - 2x + 3$
   a. Find the domain of the function.
   b. Determine the discontinuity points and behavior of the function there.
   c. Find the intercepts with the coordinate axes.
   d. Determine where the function is increasing.
   e. Determine the concavities of the function.
   f. Indicate the end behavior of the function.
   g. Sketch its graph.

2. For the function $f(x) = \frac{x + 1}{x - 1}$ repeat the same steps as in (1).
Interpretation of the second derivative as the rate of change of the first derivative

- If the function is concave up the rate of change is increasing
- If the function is concave down, the rate of change is decreasing.

Distance, Velocity and Acceleration

- \( y = s(t) \) position of an object after \( t \) seconds
- Velocity \( v(t) = \frac{ds}{dt} = s'(t) \)
- Acceleration \( a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = s''(t) \)

Velocity and Acceleration from the Graph of a Distance Function

During an experiment a mouse is placed at the end of a tunnel. During the mouse search to find the exit to the tunnel he moves away and toward the center. The graph below shows the distance, velocity, and acceleration, of a mouse that is moving inside the tunnel. Consider only times \( t \geq 0 \).

- Which graph represents the distance, which the velocity, and which the acceleration? Explain.
- On which intervals is the mouse moving away from the center of the tunnel? What is the sign of the velocity on those intervals?
- At which times is the mouse moving toward the center? What is the sign of the velocity at those times? Why?
- At which times is the mouse changing direction?
- At which time is its acceleration positive? Interpret it in terms of the distance function? How about in terms of the velocity function?
- Find the critical points of the distance function. Are they any local maximum? Minimum?
- Find the critical points of the velocity function. What is the meaning of those in terms of the distance function?
- Suppose the equation of the distance is given by the expression \( d(t) = 100(t - 2)^2 \). Algebraically find (use technology to find the first and second derivative if needed):
  - The times when the function is increasing.
  - The times when the function is concave down.
  - The times when the acceleration is zero.
  - The times when the mouse is at the starting point?
  - The times when the mouse changes direction.

### The Second Derivative and Local Max/Min

In some opportunities we can determine whether a point is a local maximum or local minimum by looking at the second derivative at that point.

So far we know that whenever \( f''(x) > 0 \) the function is concave up and whenever \( f''(x) < 0 \) the function is concave down. Hence, if we have a critical point of a function and the second derivative is either positive or negative, immediately we can guarantee the point is either a local maximum or minimum depending on the concavity. However, when the second derivative is zero it could be either local max or local min.

For instance, let’s look at the function \( f(x) = x^4 \); this function is concave up. Its only critical point is at \( x = 0 \), solution to \( f'(x) = 0, 4x^3 = 0 \). This guarantees that the point is a local minimum. Its second derivative is \( f''(x) = 12x^2, f''(0) = 0 \).

On the other hand, the function \( f(x) = -x^4 \) is a concave down function. Its only critical point is at \( x = 0 \), solution to \( f'(x) = 0, -4x^3 = 0 \). This guarantees that the point is a local maximum. Its second derivative is \( f''(x) = -12x^2, f''(0) = 0 \).

This example illustrates the situation that when the second derivative is zero at a critical point, the critical point could become a local maximum or minimum. So to decide which one it is, one can not use the second derivative.

### Second Derivative Test

Let \( f(x) \) be with \( x = a \) as a critical point:

1. If \( f''(a) > 0 \), there is a local minimum at \( x = a \).
2. If \( f''(a) < 0 \), there is a local maximum at \( x = a \).
3. If \( f''(a) = 0 \), it could be a local max or local min. For that reason the test is inconclusive and we need to go back to the first derivative test.

Need to include examples for the second derivative
EXERCISE 4
1. Sketch a graph of a function where a critical point is an inflection point.
2. Sketch a graph of a function where a critical point is not an inflection point.
3. Sketch a graph of a function where an inflection point is not a critical point.
4. Sketch a graph of a smooth function with one critical point that is an inflection point, and one inflection point that is not a critical point. Are there any local max/min values?