Purpose:

This lab has three purposes:

- You will plot the graph of Taylor polynomials to see how well they approximate a function.
- You will use the graphic representation to estimate the error between the function and the Taylor Polynomial.
- In order to approximate a definite integral we will integrate the Taylor polynomial to get an approximation in the case of the standard normal distribution.

Problem #1: Given the function \( f(x) = e^x \), we're going to study the first 5 terms of the Taylor Polynomial at \( a = 0 \) (thus, a Maclaurin polynomial).

- Enter the function \( e^x \) as \( Y_1 \). Also enter the functions \( e^x - 0.1 \) as \( Y_2 \) and \( e^x + 0.1 \) as \( Y_3 \). Go to the Window button and set the bounds on \( x \) from \(-3\) to \( 3\). Use your own judgment for \( y \). Then plot these three functions.

- Look at the resulting graph, and choose the "Zoom In" tool (it's #2 on the Zoom menu). When you pick "Zoom In", you get a little flashing cursor on the screen. Use the down arrow to move the cursor so that it's more or less on top of the graphs at \( x = 0 \), and hit Enter. Do this another time or two until you can clearly see three different curves. The middle one is your original \( e^x \), and the two curves around it are little neighborhoods of \( +/- 0.1 \). In the instructions below, we're going to call this neighborhood the "tube". We'll see why it's useful in a bullet or two.

- Next, find the Taylor polynomial up to degree 4 for \( f(x) = e^x \) by hand (you can probably just get this from lecture notes). Using this, enter the first-degree part of this polynomial (i.e., \( 1 + x \)) as \( Y_4 \). Reset the bounds on the graph to \( x \) between \(-3\) and \( 3\), and plot it. The plot of the straight line approximation should be added to the graph you've already started above. Now, using the Zoom In and Zoom Out tool, try to find where the straight line exits the little tube around \( e^x \) somewhere to the right of \( x = 0 \). We've got a picture on the next page of what it should look like from Matlab:

When this gets copied, you won't be able to see color, but you can see the red line exiting the tube at around \( x = 0.42 \). You can zoom in farther if you want more accuracy.
• Now try it yourself. Reset $x$ to –3 to 3, refresh the graph, and try to discover where the straight line exits the tube for $x < 0$. For your lab report, jot down the $x$-value where this happens. You should have at least two decimal place accuracy.

• What is this supposed to tell you? As long as the straight line is inside the tube, then the $y$-values of the linear approximation are within 0.1 of the true value; that is, you'd have one decimal place accuracy for these $x$'s. Said a different way, if the answer you found on the last bullet was –0.59 (it won't be), then for all $x$-values between $x = –0.59$ and $x = 0.42$, $t_1(x) = 1 + x$ is an approximation of $e^x$ accurate to one decimal place.

• Go back to Y= and clear Y4. This time enter the second degree Taylor polynomial for $e^x$. Start with a fresh graph, and look to see where the second degree polynomial leaves the tube. Write down your two $x$-values to two decimal place accuracy.

• Repeat for $t_3$ and $t_4$, the third and fourth degree Taylor polynomials for $e^x$.

• To turn in for Problem #1: a table with the degree $n$ of the polynomial, and the corresponding $x$-values below and above 0 where the function $t_n$ left the tube. Also, sketch either of your graphs for $t_4$ like we did above, so that you can see clearly where the approximation leaves the tube.
Problem #2: Do the same for the function \( f(x) = \sin(x) \) at \( a = \pi / 2 \). Note that, since your center is not \( a = 0 \), the Taylor series for \( \sin(x) \) won't be the same one you saw in class. Also, use a tube of \( +/- 0.01 \) instead of 0.1 as you did above. Turn in the same sort of thing you did for Problem #1.

Problem #3: (This one's probably done best mostly on paper except for the graph at the very end. Just turn in your work for this problem.) The probability function of the standard normal distribution is given by \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \). (We've got it right this time.)

The cumulative distribution function is defined by \( F(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \). We talked about this integral in the lab two weeks ago, where you were asked to use Simpson's rule to approximate this integral because is not a primitive function (antiderivative) for \( f(x) \), and so you can't use the Fundamental Theorem of Calculus.

A different option for finding \( F(a) \) is to approximate \( f(x) \) by its Taylor polynomial, and then do a term by term integration.

Here's an example with a simpler function, \( f(x) = \cos x \). The exact integral of \( g \) from \( x = 0 \) to \( x = 1 \) is \( \int_{0}^{1} \cos x \, dx = \sin 1 - \sin 0 = 0.8415 \). Suppose instead we chose to integrate the fourth degree Taylor polynomial for cosine:

\[
\int_{0}^{1} \frac{1}{2!} x^2 + \frac{1}{4!} x^4 \, dx = 1 - \frac{1}{3!} + \frac{1}{5!} = 0.8417.
\]

Thus, we get at least three decimal place accuracy in this example by integrating the Taylor polynomial instead of the original function.

Now do this yourself with \( f(x) \). Find the 4th degree Taylor polynomial for \( f \) by hand. (It's easier to just ignore the \( \frac{1}{\sqrt{2\pi}} \) at first, find the Taylor polynomial for \( e^{-x^2/2} \), then multiply that result by \( \frac{1}{\sqrt{2\pi}} \).) Then integrate the resulting polynomial from \( x = 0 \) to \( x = 1 \). Recall from the last lab that \( F(a) = \frac{1}{2} + \int_{0}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \), so if you add 1/2 to your integral, you should get something pretty close to \( F(1) = 0.8413 \). Also estimate \( F(2) = 0.9772 \) (your answer here should be a lousy estimate). Plot the standard normal function \( f \) and its 4th degree Taylor polynomial from \( x = 0 \) to \( x = 2 \). Use the graph to explain in your own words why the estimate for \( F(1) \) wasn't bad, but the estimate for \( F(2) \) was lousy.

Problem 4 (continuation of #3, extra credit): Use your graph from #3 to find the greatest difference between the standard normal function \( f \) and its 4th degree Taylor polynomial between \( x = 0 \) and \( x = 1 \). Use this difference to come up with an upper bound on the error for the estimation of \( F(x) \) for \( 0 \leq x \leq 1 \).