Laboratory I.9
Applications of the Derivative

Goals
• The student will determine intervals where a function is increasing or decreasing using the first derivative.
• The student will find local minima and maxima of functions.
• The student will determine the intervals where a function is concave up or concave down.
• The student will use the derivative of a function to solve an application problem.

Before the Lab
The material in this lab is essentially the same as that learned in Chapter 4, sections 3 and 7 in the text. This lab gives you a chance to practice the ideas without worrying about the algebra.

In the Lab (90 pts)

1. As you work through this problem, fill in your answers in “Table 1” in the After the Lab part.

(a) **Author** and **Plot** \( f(x) := \frac{2x^2 + x - 1}{2x^2 + 5x + 4} \). Use the **Trace** function of Derive to find the approximate coordinates of the local minimum and the local maximum that you see. Put the \((x, y)\) coordinates of these two points in Table 1.

(b) Recall from class that a function is increasing where its derivative is positive. We can get Derive to find where this is true for \( f(x) \). Use **Calculus Differentiate** \( f'(x) \) to find the derivative, then **Author** \( (\text{the derivative you just found}) > 0 \). To avoid having to type the derivative in, make sure the derivative is highlighted when you open the **Author** box, then use the F3 key to copy it in. Now **Author** \( \text{solve}(\text{the inequality you just authored}, x) \) to solve the resulting inequality. Repeat for \( f''(x) < 0 \) to see where the function is decreasing, and fill in the appropriate lines in Table 1.

(c) To find the minimum and maximum, **Author** \( (\text{your derivative}) = 0 \), then **Solve** and **Approximate** the results. Use \( f(x) \) to obtain the \( y \) values, and fill in the appropriate places in Table 1.

(d) Finally, the concavity of the function \( f(x) \) depends on the second derivative: where \( f''(x) > 0 \), the function is concave up; where \( f''(x) < 0 \), the function is concave down. Go through steps (b) and (c) again with the second derivative to find where the function is concave up, concave down, and the points where the concavity changes.

2. Repeat the above for \( f(x) := x^3 - 3x^2 + 3x - 5 \). Put the results in Table 2.

3. A customer wants to know the cost of a water tank that will hold 800 gallons. Your firm can make the tank from a rectangular piece of metal that you will roll into a cylinder and two circular ends that will be welded to the cylinder. The rectangular piece is made of a malleable alloy that cost $2.36 per square
foot, but the circular ends can be made of a cheaper metal costing $1.44 per square foot. Welding costs $1.20 per foot. In general the firm adds 23% of the total cost to get the selling price of any tank they build. (7.5 gal = 1 ft³)

(a) Find the height (h) and radius (r) of the tank that would hold 800 gallons and would cost the least.
   i) Find a function C that gives the cost of making a tank of radius r and height h.
   ii) Write an equation that relates the volume of the tank, 800 gallons, to an unknown radius r and height h.
   iii) Solve the above equation for one of the variables, h, and substitute for that variable in the function C. Now C will depend on only one variable, r.
   iv) Minimize the cost C using what you learned in parts 1, 2 above.

(b) What would be the selling price of this tank?

(c) If the customer who wants to buy the tank has only 225 dollars and the firm need to keep a 23% profit margin on any tank they build, what is the maximum volume of the tank the firm can build for 225 dollars.

If you need some help to find answers to these questions follow the steps given below carefully.

To find a solution we will follow the steps outlined in (i) through (iv). You will find it helpful if some of the quantities we work with are expressed as words rather than single letters. For example, we will later write TotalCost instead of T. To make Derive accept such names, set the Input Mode to Word rather than Character.

In Derive, choose Options::Mode Settings. In that dialog box, click on the Input Tab and choose “Word.” You might want to also select “Sensitive.” (Who wants to be an insensitive character, anyway?)

Let’s get started to find an equation that relates cost (C) and radius (r).

Cost calculation.
It will be helpful to calculate the cost of the different parts of the tank and the welding separately.

Cost of the each circular piece = area * cost
= \( (\pi r^2) \times 1.44 \)

In Derive, Author CircularCostPiece:= \( \pi r^2 \times 1.44 \) (note the colon)

Cost of the rectangular piece = area * cost
= \( (2 \pi r h) \times 2.36 \)

In Derive, Author RectangularPieceCost:= \( 2 \pi r h \times 2.36 \) (note the colon)

Cost for welding = cost of welding rec. piece to form a cylinder + 2 * cost of welding circular piece to the cylinder
= \( (h \times 1.20) + 2 \times (2 \pi r \times 1.20) \)

In Derive, Author WeldingCost:= \( h \times 1.20 + 2 \times (2 \pi r \times 1.20) \) (note the colon)

To find the total cost of the tank, including parts and assembly, add the three costs.

In Derive Author TotalCost = 2*CircularCostPiece + RectangularPieceCost + WeldingCost (note the lack of a colon)

Derive’s response won’t be much help. Highlight the right side of the last equation and Simplify it to get some satisfaction from Derive. Set that result aside for a while.

**Bring In the Volume**

We need to get back to the equation relating \( V, r \) and \( h \). Of course, Volume of the cylinder = area of a circular piece * height of the cylinder. Author the following equation in Derive.

\[ V = \pi r^2 h \] (note the lack of a colon)

Let us call this the volume equation.

Since \( h \) and \( r \) are in feet we need to convert 800 gal to cubic feet. If 7.5 gal = 1 ft\(^3\), then

\[ 800 \text{ gal} = \ldots \ldots \ldots \ldots \ldots \ldots \text{ft}^3 \]

In Derive, highlight the volume equation substitute \( V = “\text{what you got above”} \) and solve the result for \( h \).

You should get an expression for \( h \) that only depends on \( r \). Highlight that portion of the equation (not the \( h = \) part). Copy that portion (Edit::Copy). We will substitute what you copied into the expression for the total cost.

In Derive, go back to the simplified version of the total cost and highlight it. Select the Variable Substitution command from the Simplify menu. Select the variable \( h \) and paste in what you copied to
substitute it for \( h \). The result will be an expression for the cost of the tank that only depends on \( r \). You can make a function of this expression. Copy it and then Author \( C(r) := \) and paste in the expression.

Analyzing this function \( C(r) \) as for problems 2 and 3 is what you need to do to find the radius of the tank of minimum cost. Of course, you will have find the value for \( h \).

Plot this expression. If you cannot see the graph go to Set:Plot Range:Minimum/Maximum and set the horizontal scale to -20:20:8 and vertical scale to -800:800:8. Can you see how the cost varies as the radius grows? Since we want to minimize the cost, pick the radius that corresponds to the minimum cost. As you may realize, it is difficult to get the exact radius from this graph. Do you know how to find the exact value of \( r \) that gives you the minimum cost? Well, find the derivative of the function and plot it on the same graph. You need to click on the expression until the right hand side of it is highlighted. Then click on the derivative button to find the derivative of the highlighted section. At a minimum or a maximum the first derivative should cross the x-axis. Make this expression equal to zero and solve it numerically with an upper bound of 10 and lower bound of 0. When you find the value of \( r \), substitute it in the expression that contains only \( h \) and \( r \) and find the value of \( h \).

To find the answer for b, simply substitute the values of \( h \) and \( r \) in the total cost expression. Don’t forget to add 23% of the total cost to get the selling price. \( \text{TotalCost} + \text{TotalCost} \times 0.23 = \text{SellingPrice} \).

To find the answers for (c) we have to work backward. First of all, if you want to sell the tank for $225, what should be the total cost of the tank? Remember, 225 is the price you get after adding 23% to the total cost. Solving the above equation for the TotalCost can find the selling price.

Total cost of building the tank is ……………….. (When you have this value add 23% to it. If you are not getting 225, then it is not correct)

Substitute the value you got in the above calculation for TotalCost in the cost equation. Now you have an expression that contains only \( h \) and \( r \). Then solve this expression for \( h \) and express it in terms of \( r \). Volume expression has \( h \) and \( r \). Replace \( h \) in the volume expression with what you got for \( h \). Now you have an expression of \( V \) and \( r \). Plot this expression and observe how the volume varies with radius. Using the same method find the radius that would give you the maximum volume. Make sure to highlight the right side of the expression before finding the derivative and when solving it for zero set the bounds to 1 and 10. Then find the maximum volume of the tank in gallons the firm can build for what she can afford.

Ready for the Lab?

Again, there are no “Ready for Lab?” questions. You should turn in the results for #4, Tables 1 and 2, answers to the “After the Lab” questions and hard copy of Derive algebra window.

After the Lab (10 pts)

1. Looking at your work from “In the Lab” exercises 1 & 2.
   (a) If you have a maximum, what seems to be true about the concavity at that maximum? What about a minimum?
   (b) How would you identify a place where the derivative is 0, but there’s no maximum or minimum there?
2. Answer the questions from problem 3:
   (a) Find the height (h) and radius (r) of the tank that would hold 800 gallons and would cost the least.

   (b) What would be the selling price of this tank?

   (c) If the customer who wants to buy the tank has only 225 dollars and the firm need to keep a 23% profit margin on any tank they build, what is the maximum volume of the tank the firm can build for $225?
3. Tables to turn in with the Derive printouts.

**Table 1: \( f(x) = \frac{2x^2 + x - 1}{2x^2 + 5x + 4} \)**

<table>
<thead>
<tr>
<th>Approximate min and max from the graph. Identify which is which. Give your answers in the form “Max at (x,y)...”</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Interval(s) on which the graph is increasing</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Interval(s) on which the graph is decreasing</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Predict from the first derivative at which point a minimum, a maximum or a point of inflection is found, Write down the x-values</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Interval(s) on which the graph is concave up</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Interval(s) on which the graph is concave down</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Point(s) where the concavity changes</strong></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2: \( f(x) = x^3 - 3x^2 + 3x - 5 \)**

| **Interval(s) on which the graph is increasing** |  |
| **Interval(s) on which the graph is decreasing** |  |
| **Predict from the first derivative at which point a minimum, a maximum or a point of inflection is found** |  |
| **Interval(s) on which the graph is concave up** |  |
| **Interval(s) on which the graph is concave down** |  |
| **Point(s) where the concavity changes** |  |