Laboratory I.8
Newton’s Method

Goals:
• The student will learn how to solve equations using Newton’s Method.
• The student will get an introduction to iteration and dynamical systems.

Before the Lab:

Newton’s Method is a technique which uses derivatives to solve equations that
you can’t solve by hand (algebraically). An example is \( \cos x = x \). There are no algebraic
tools which allow you to isolate \( x \) on only one side of that equation.

We’ll have you solve that particular equation in the lab; we’re going to work as
our example a simpler equation that we can solve by hand so that we know what we’re
doing. We’ll do \( x^2 = 2 \), and in particular we’ll pretend we don’t know the positive
solution, \( x = \sqrt{2} \).

Solving \( x^2 = 2 \) is the same as solving \( x^2 - 2 = 0 \), so let’s graph \( y = x^2 - 2 \) and look
where it crosses the \( x \)-axis:

You can see the solution is just to the right of \( x = 1.4 \).

Newton’s Method works by taking an initial
guess and improving it. Let’s use as our initial
guess \( x_0 = 2 \). We’re going to add a
tangent line to \( y = x^2 - 2 \) to the picture:

The tangent line hits the \( x \)-axis at \( x = 1.5 \). Notice
that this is much closer to the exact solution than
our original guess, \( x_0 = 2 \). This, in essence, is
Newton’s Method: the tangent line to the curve at
a point will hit the \( x \)-axis closer to the exact root
than the starting point.

Let’s check this with another picture. Our
original guess of \( x_0 = 2 \) was “improved” to a guess of \( x_1 = 1.5 \). Let’s draw the tangent line
to \( y = x^2 - 2 \) at \( x = 1.5 \) and see what we get:
The new tangent line is so close to the function that we can hardly tell them apart. The intersection of the new tangent line with the x-axis is at $x_2 = 1.4167$; the exact value of the solution is $x = 1.4142$, so after just two iterations of Newton’s method, we have reduced our error to 0.0025. If we needed more decimal places of accuracy, we could continue, finding the tangent line for our current guess, $x_2 = 1.4167$.

So, this gives us **Newton’s Method (verbal/graphical version):**

To find a root of $f(x) = 0$, start with an initial guess $x_0$. Find the tangent line to $f(x)$ at $x = x_0$, and from that find where the tangent line hits the x-axis. That point of intersection is $x_1$. Continue following the same process from $x_1$ to get $x_2$, $x_3$, . . . until some estimate $x_n$ is “close enough” to quit.

To use Derive (or any other computer tool) with Newton’s Method, we have to convert this verbal/graphical process to an algebraic one. So, let’s solve for $x_1$ as a function of $x_0$.

- Using the point-slope form, the equation for the tangent line to $f'(x)$ at $x = x_0$ is

  $$y - f(x_0) = f'(x_0)(x - x_0)$$

- $x_1$ is the point at which this line hits the x-axis, so we plug in $x_1$ for $x$ and 0 in for $y$:

  $$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

- We solve this equation for $x_1$

  $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

So this gives us **Newton’s Method (algebraic version): to find a root of $f(x) = 0$, start with an initial guess $x_0$. Find a sequence of new guesses using the rule**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to get new guesses from the previous guess. Continue until some estimate $x_n$ is “close enough” to quit.

The definition of “close enough” varies according to different uses of Newton’s Method; we will use the following: **since we want $f(x) = 0$, stop when $|f(x_n)| < 0.01$.**

In The Lab (90 pts)

0. The most efficient way to get Derive to compute stuff for Newton’s Method is to
Author: Variable Domain \(x_0\) (This means choose Variable Domain from the Author menu; ‘All’ should be selected. Press OK). Otherwise, Derive wants \(x_0\) to be \(x\) multiplied by 0

\[
\text{Author}\ \text{Newton}(u, x, x_0, n) := \text{iterates}(x-u/d\!f(u, x), x, x_0, n)
\]

What this does is produce the first \(n\) guesses after your initial guess \(x_0\). \(u\) stands for your function, \(x\) is the name of your variable, \(x_0\) is your initial guess, and \(n\) is the number of guesses you want. For example, to reproduce the example in “Before the Lab”, you would first Author the above, then

\[
\text{Author}\ f(x):= x^2 -2 \text{ and } \text{Newton}(f(x), x, 2, 3), \text{press enter and then approximate the result. (Use the approximate(≈) sign in the Author box). Now look at the answer:}
\]

\[\{2, 1.5, 1.416666666, 1.414215686\}\]

2 is the initial guess and 1.5, 1.4166….. , and 1.4142….. are the 3 consecutive \(x\) intercepts of the tangent lines.

Now use Derive (or your calculator) to see if \(|f(1.4142)|\) is indeed less than 0.01, or whether you’d need to keep going.

1. Find all solutions for each of the following equations. Start by re-writing the equation so that one side is 0, in other words if your initial function is \(x^2 = 2x+3\) , then rewrite as \(x^2 -2x -3 =0\) and then define \(f(x):= x^2 - 2x - 3\) and graph the resulting function to see how many roots there are and a nearby starting point for each root. Then use Newton’s method with your initial guesses to see what the roots are. Again, stop when \(|f(x_n)| < 0.01\).

   a) \(x^3 = 3\)
   b) \(e^x = 3x\)
   c) \(\cos x = x\)

2. Newton’s Method is pretty much the fastest way there is to solve equations numerically, and as such it’s frequently used in industrial and applied situations when folks need to solve an equation. However, Newton’s Method is not perfect: sometimes a bad initial guess can lead it astray; sometimes it just doesn’t work at all.

   a) Consider finding the roots of \(4xe^{-x} = 1\). What happens if your initial guess is \(x_0 = 1\)?

   Explain what goes wrong either with a picture or algebraically. Pick a better initial guess for each root and find them.

   Note: Before working on part (b), Author Branch :=Real and press enter.

b) Consider finding the roots of \((x - 2)^{1/3} = 0\). (OK, I know the root is \(x = 2\), but let’s pretend we don’t know for another example of what goes wrong.) Starting with \(x_0 = 1\), what goes wrong when you try to use Newton’s Method?
Experiment to see if you can find any other value of $x_0$ that gets you close to $x = 2$. Copy a graph of this function with the $x$-scale in the $-10$ to $10$ range, and sketch at least two tangent lines you get starting from an initial guess of $x_0 = 1$. Mark $x_0$, $x_1$ and $x_2$ on your graph.

3. What happens if you try to use Newton’s Method when there’s no root at all? For example, try to find a root of $x^2 + 1 = 0$ with several initial guesses. Be sure to use $n = 10$ or so.

Ready for Lab?

None

After the Lab (10 pts)

(1) Fill in the chart below for the functions from problem 1 of the In the Lab part above; $x_0$ is your initial guess, $n$ is the number of iterations done, $x_n$ is the approximation to the root, in other words value of $x$ that made $|f(x)| < 0.01$ and in the last column the value of $f(x_n)$. If there is more than one root, use different starting points to find the different roots. (3pts)

<table>
<thead>
<tr>
<th>Solution for problem:</th>
<th>$n$</th>
<th>$x_0$</th>
<th>$x_n$</th>
<th>$f(x_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(a)</td>
<td></td>
<td></td>
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<tr>
<td>1(b)</td>
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<tr>
<td>1(c)</td>
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</tbody>
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(2) (a) Describe your graph from #2b (2 pts)  
(b) Provide written responses to the questions in #2a and #2b (3 pts)

(3) Explain from #3 what happens in general when you try to find a root for $x^2 + 1 = 0$ (2 pts)