Laboratory I.7
Linking Up with the Chain Rule

Goal
• The student will figure out the Chain Rule for certain example functions.

Before the Lab

The Chain Rule is the derivative rule which accounts for function composition. For example, consider the function \( f(x) = \sin(x^2) \). You know that the derivative of \( \sin x \) is \( \cos x \), and that the derivative of \( x^2 \) is \( 2x \). The Chain Rule is what tells you how to put these two individual derivatives together to get the derivative of \( \sin(x^2) \).

In the Lab (90 pts )

1. The Power Rule for computing derivatives says that \( \frac{d}{dx} x^2 = 2x \). So, we are going to assume that the derivative of \( \frac{d}{dx} (f(x))^2 \) contains \( 2f(x) \), and look at examples to see if we are right. We’ll also look to see what else, if anything, is in the derivative of \( f(x)^2 \) besides \( 2f(x) \).

   A page or two down, you’ll find Table 1. Follow these instructions to fill in Table 1, and turn it in as a part of your work for this Lab. Each of the functions in column 1 is of the form \( f(x)^2 \). For each of these functions:
   (i) Author it into Derive;
   (ii) expand it in Derive (use Simplify Expand . . . , then expand in the variable \( x \))’
   (iii) use Calculus Differentiate to find the derivative function of the expanded form;
   (iv) factor the derivative (use Simplify Factor . . . in \( x \))
   (v) write the result in the table as \( 2f(x)k(x) \), where \( k(x) \) is whatever is left from the derivative after you take out the \( 2f(x) \).

   You should do the example on the second row before doing the rest of the table to make sure you understand the instructions. Note that \( f(x) \) of the example in the second row is \( (ax+b) \).

2. In the last example, we used different \( f(x) \)’s, but always stuck with the exponent 2. Here, we’ll use the same function each time \( f(x) = x^2 - 3x \), but different exponents. We’re also going to be limiting ourselves to studying what happens at a particular point, \( x = -1 \).
Find Table 2, and follow these instructions to fill it in. Again, we’ve done one as an example for you to check, and you should turn this in with the rest of your work on this lab.

(i) **Author** \( f(x) := x^2 - 3x \)
(ii) **Author** \( n f(x)^{n-1} \)
(iii) Use Simplify:Variable Substitution to evaluate the function \( n f(x)^{n-1} \) at \( x = -1 \) and \( n \) from the first column of Table 1.
(iv) **Author** \((f(x+h)-f(x))/h\) and use Simplify:Variable substitution to estimate the derivative at \( x = -1, h=0.001 \) and \( n \) from the first column.
(v) either by hand or by Derive, find \( f'(\cdot) \).

3. Finally, we do one more table using the sine function. Again we are looking for the relationship between the derivative of \( \sin[f(x)] \), \( \cos[f(x)] \) (because cosine is the derivative of sine), and \( f'(x) \). Do the following steps to fill out Table 3 in a similar fashion to Table 2.

(i) **Author** \( f(x) := 2x \)
(ii) **Author** \( \cos(f(x)) \); Use this expression to fill in column 2.
(iii) **Author** \( (\sin(f(x+h)) - \sin(f(x)))/h \); Use this to approximate the derivative of \( \sin(f(x)) \), that is to fill in column 3.
(iv) either by hand or by Derive, find \( f''(3) \), for column 4.
Since \( f(x) = 2x \), these steps are helpful to understand where those numbers in row 2, Table 3 came from. To fill out the rest of the table change \( f(x) \) to the appropriate function given in column one and follow the steps (ii) – (iv).

**Ready for Lab**
Sadley enough, there are no “Ready for Lab” questions this week. To turn in: you should turn in the completed tables starting below, as well as answers to the questions in “After the Lab.”

**Important:**
Each of the three tables on the next and subsequent page follow the same pattern. The first row contains the headings for the columns. The second row is an example of how a completed row looks like for a specific example. The remaining rows are to be filled in by you according to the entry in the leftmost column.
### Table 1

<table>
<thead>
<tr>
<th>$f(x)^2$</th>
<th>$2f(x)k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(ax + b)^2$</td>
<td>$2(ax + b)(a)$</td>
</tr>
<tr>
<td>$(ax^2 + bx + c)^2$</td>
<td></td>
</tr>
<tr>
<td>$(ax^3 + bx^2 + cx + d)^2$</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2

Consider the function $f(x) = x^2 - 3x$. You will work with the function $(f(x))^n$ for each of the values of $n$ given in the first column.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Value of $nf(x)^{n-1}$ at $x = -1$</th>
<th>Approx value of $\frac{d}{dx} (f(x))^n$ at $x = -1$</th>
<th>Value of $f'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2((-1)^2 - 3(-1)) = 8$</td>
<td>$(f(-1+.001)^2-f(-1)^2) / .001 = -39.9659$</td>
<td>-5</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. (a) In Table 1, what relationship does the function \( k(x) \) bear to the original function \( f(x) \)?

(b) After you figure out their relationship, write down the resulting rule from Table 1:

\[
\frac{dx}{d(f'(x) = \text{______________________________}}
\]

(c) Without using Derive, use your rule above to find \( \frac{dx}{d} \) at \( x = 8 \). (Show all work)

2. (a) In Table 2, pick a row and look at the numbers in each column. What is the relationship between the three columns you filled in? After analyzing data you should be able to come up with a rule that works for all the four rows.

(b) Write down the resulting rule from Table 2: (Here I want you to write down a some sort of expression, rather than numbers from table 2. All you need to write down this expression can be found in the first row of table 2)

\[
\frac{dx}{d} (f'(x)^n) = \text{______________________________}
\]
(c) Without using Derive, use your rule above to find
\[
\frac{d}{dx} (x^3 - 2x^2 - 2x + 6)^3 \text{ at } x = -4. \text{ (Show all work)}
\]

3. (a) In Table 3, what is the relationship between the three columns you filled in?
(b) Write down the resulting rule from Table 3. (similar to question 2.)

\[
\frac{d}{dx} \sin (f(x)) = \text{__________________________}
\]

(c) Without using Derive, use your rule above to find
\[
\frac{d}{dx} \sin (x^2) \text{ at } x = \pi. \text{ (Show all work)}
\]

4. Combine rules 1-3 above:
\[
\frac{d}{dx} g(f(x)) = \text{__________________________}
\]